

Exotic Non-relativistic String

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We construct a classical non-relativistic string model in 3+1 dimensions. The model contains a spurion tensor field that is responsible for the non-commutative structure of the model. Under double dimensional reduction the model reduces to the exotic non-relativistic particle in 2+1 dimensions.

I. INTRODUCTION

The free massive non-relativistic particle in D dimensions has as a symmetry group, the Galilei group, with a central extension associated to the mass of the particle. Instead, in 2+1 dimensions, there is a two-fold central extension [1, 2, 3], where the second central element is interpreted as a non-commutative parameter. These central extensions are related to a

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nontrivial Eilenberg-Chevalley cohomology of degree two. In fact every closed invariant 2-form that locally is the differential of a 1-form, which is not invariant, produces an extension of the algebra, see for example [4, 5, 6].

The exotic (2+1)D Galilei symmetry has appeared in the context of non-commutative geometry and condensed matter physics [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. It is a symmetry of a free relativistic particle in a non-commutative plane in a special non-relativistic limit [9, 11, 12].

Instead, if one consider the non-relativistic limit of a relativistic particle in the AdS in 2+1 dimensions, one finds a system with an exotic Newton-Hooke symmetry [18]. The system has three essentially different phases [19], depending on the values of the two central charges, which are present in the model. The subcritical and supercritical phases (describing 2D isotropic ordinary and exotic oscillators) are separated by the critical phase (one-mode oscillator).

Non-relativistic extended objects has been recently considered in the literature. In particular, non-relativistic string theory [20] (see also [21]) in flat space is a consistent decoupled sector of the bosonic string theory, whose worldsheet, in the conformal field theory description [20], possesses the appropriate Galilean symmetry. Non-relativistic string theory can be derived as a certain decoupling limit of the original relativistic theory, even though the theory can be written down without any reference to the original parent theory. The basic idea behind the decoupling limit is to take a particular *non-relativistic* limit, in such a way that only states satisfying a Galilean invariant dispersion relation have finite energy, while the rest of the states decouples.

The action of non-relativistic bosonic string theory, for a flat D dimensional space-time, can be obtained with the method of non-linear realizations [22] as a Wess-Zumino (WS) term of the appropriate (string) Galilean group [23, 24]. The extended algebra has non-central elements that transform non-trivially under the stability group.

Particle models can be obtained by a double-dimensional reduction of strings models. It is natural to ask if it is possible to construct an exotic non-relativistic string in 3+1 dimensions, with other additional extended charges, such that by reduction one could obtain the exotic particle models appeared in the literature [7, 8, 11, 12]. To this end we construct the classical action for exotic Galilean non-relativistic strings, using the method of non-linear realizations. We first construct the exotic string Galilei algebra in 3+1 dimensions. This

algebra contains new extra extended generators. As we will see, the parameters associated to these new extensions are related to non-commutative parameters. The string action is constructed from a 2-form, which is the product of two invariant Maurer-Cartan (MC) 1-forms of the exotic extended string Galilei algebra. The parameters of the string action are the tension and a tensor parameter (spurion), generalizing the magnetic field in the case of a particle.

We perform a classical Hamiltonian analysis of the model. The hamiltonian turns out to be non-local in the original variables. If we introduce an appropriate change of variables, we can eliminate the non-locality, ending with a theory formally equivalent to the ordinary non-relativistic one for particular boundary conditions. Therefore, in this last case, the symmetry group of the model in the new variables is the non-exotic Galilei group.

The organization of the paper is as follows. In Section II we consider the extended exotic Galilei algebra. In Section III we introduce the string action and in Section IV we study the dynamics in the static gauge, showing that there are second class constraints in the model, then we proceed to evaluate the Dirac brackets. This analysis shows that the Dirac brackets of the position variables are not zero, giving rise to a non-commutative theory. In Section V we show that it is possible, through a change of variables, to make the model equivalent to the non relativistic string model, only for particular choices of the boundary conditions. We conclude with an outlook and a discussion in Section VI.

II. EXOTIC EXTENDED GALILEI STRING ALGEBRA

Let us consider a non-relativistic string in a D dimensional flat space time.

We will denote by x^0, x^1 and x^2, \dots, x^{D-1} the longitudinal (along the string) and transverse coordinates respectively. The symmetry algebra of this system is the extended Galilei algebra, G [23] [24], which is given by

$$\begin{aligned} [P_\mu, P_\nu] &= 0, & [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= 0, \end{aligned} \tag{1}$$

$$\begin{aligned} [M_{ab}, P_c] &= i(\eta_{ac}P_b - \eta_{bc}P_a), \\ [M_{ab}, M_{cd}] &= i(\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}), \end{aligned} \tag{2}$$

$$\begin{aligned}
[P_\mu, P_a] &= 0, & [M_{\mu a}, P_\nu] &= i\eta_{\mu\nu} P_a, \\
[M_{\mu\nu}, M_{\rho c}] &= i(\eta_{\mu\rho} M_{\nu c} - \eta_{\nu\rho} M_{\mu c}), & [M_{\mu a}, M_{bc}] &= -i(\eta_{ab} M_{\mu c} - \eta_{ac} M_{\mu b}),
\end{aligned} \tag{3}$$

$$[P_a, M_{\nu b}] = i\eta_{ab} Z_\nu, \quad [M_{\mu a}, M_{\nu b}] = i\eta_{ab} \epsilon_{\mu\nu} Z, \tag{4}$$

and

$$[P_\mu, Z_\nu] = 0,$$

$$[P_\mu, Z] = +i\eta_{\mu\nu} \epsilon^{\nu\lambda} Z_\lambda, \tag{5}$$

$$[Z_\mu, M_{\rho\sigma}] = -i\eta_{\mu[\rho} Z_{\sigma]} \quad [Z, M_{\rho\sigma}] = 0. \tag{6}$$

In these expressions $P_\mu, \mu = 0, 1$, are the longitudinal translation generators, $P_a, a = 2, \dots, D-1$ are the transverse ones, $M_{\mu\nu}, M_{\mu a}, M_{ab}$ are the generators of the boost transformation along the longitudinal direction, generalized boosts and rotations in the transverse space and Z, Z_μ are the extended elements. Our conventions for the metric tensor are: $\eta_{\mu\nu} = -, +, \eta_{ab} = +, +, \dots, +$. $[\mu\nu]$ means antisymmetry in the interchange of the two indices $\mu\nu$. The Levi-Civita tensor is defined with $\epsilon^{01} = +1$.

Observe that the commutator of $M_{\mu\nu}$ and $M_{\rho\sigma}$ is zero. This is due to the fact that in 2 dimensions ($\mu, \nu = 0, 1$) there is only one generator of Lorentz transformations, and no rotations.

The action of non-relativistic string can be obtained by the method of non-linear realizations [23, 24, 25]. Let us consider the coset G/H , with the stability group H generated by $(M_{\mu\nu}, M_{ab})$.

Locally we parametrize a coset element as [28]

$$g = e^{ix^\mu P_\mu} e^{ix^a P_a} e^{iv^{\mu a} M_{\mu a}} e^{ic^\mu Z_\mu} e^{icZ}. \tag{7}$$

The (Goldstone) coordinates of the coset depend on the parameters τ, σ of the world-sheet,

The MC 1-form is given by

$$\Omega = -ig^{-1}dg = L^\mu P_\mu + L^a P_a + L^{\mu a} M_{\mu a} + \frac{1}{2}L^{\mu\nu} M_{\mu\nu} + \frac{1}{2}L^{ab} M_{ab} + L_z^\mu Z_\mu + L_z Z, \tag{8}$$

where

$$\begin{aligned}
L^\mu &= dx^\mu, & L^a &= dx^a + dx^\mu v_\mu^a, \\
L^{\mu\nu} &= 0, & L^{\mu a} &= dv^{\mu a}, & L^{ab} &= 0.
\end{aligned} \tag{9}$$

$$\begin{aligned}
L_z^\mu &= dc^\mu + dx^\nu \epsilon_\nu^\mu - v_{,a}^\mu dx^a - \frac{1}{2} dx^\nu v_{\nu a} v^{\mu a}, \\
L_z &= dc + \frac{1}{2} \epsilon_{\mu\nu} v^{\mu}{}_{,a} dv^{\nu a}.
\end{aligned} \tag{10}$$

In the case of a four dimensional space-time the extended Galilei algebra has a further extension. In fact, owing to the existence of the two dimensional Levi-Civita tensor, we can construct a non-trivial closed invariant 2-form $\Omega_2^{\mu\nu}$, that transforms as a symmetric tensor of order two under the stability group

$$\Omega_2^{(\mu\nu)} = \epsilon_{ab} L^{(\mu a} L^{\nu) b} = d(\epsilon_{ab} v^{\mu a} dv^{\nu b}). \tag{11}$$

The 1-form $\epsilon_{ab} v^{\mu a} dv^{\nu b}$ is not invariant, therefore the Eilenberg-Chevalley cohomology of degree two is non trivial [4, 5, 6]. This implies the possibility of further extending the algebra. We denote the new generators of the algebra by $\tilde{Z}_{(\mu\nu)}$, where $(\mu\nu)$ means symmetric with respect the interchange of μ and ν .

The new non vanishing commutation relations are

$$[M_{\mu a}, M_{\nu b}] = i\eta_{ab} \epsilon_{\mu\nu} Z + i\epsilon_{ab} \tilde{Z}_{(\mu\nu)}, \tag{12}$$

$$[\tilde{Z}_{(\mu\nu)}, M_{\rho\sigma}] = i\eta_{\nu[\rho} \tilde{Z}_{(\mu\sigma)}] + i\eta_{\mu[\rho} \tilde{Z}_{(\nu\sigma)}]. \tag{13}$$

In the new extended algebra the MC 1-form has one more term, given by $\frac{1}{2} \tilde{L}_z^{\mu\nu} \tilde{Z}_{(\mu\nu)}$, and the coset element is

$$g = e^{ix^\mu P_\mu} e^{ix^a P_a} e^{iv^{\mu a} M_{\mu a}} e^{ic^\mu Z_\mu} e^{ic Z} e^{i\frac{1}{2} \tilde{c}^{(\mu\nu)} \tilde{Z}_{(\mu\nu)}}, \tag{14}$$

where $\tilde{c}^{(\mu\nu)}$ are the group parameters associated to generators $\tilde{Z}_{\mu\nu}$. The new term in the MC 1-form is given by

$$\tilde{L}^{(\mu\nu)} = d\tilde{c}^{\mu\nu} + \frac{1}{2} \epsilon_{ab} v^{a(\mu} dv^{\nu)b}. \tag{15}$$

III. EXOTIC STRING ACTION

The ordinary non-relativistic string action can be written as the pullback on the world-sheet with coordinates (τ, σ) of the invariant 2-form [24]

$$\Omega_2 = \epsilon_{\mu\nu} L^\mu L_z^\nu = \epsilon_{\mu\nu} dx^\mu \left(dc^\nu - v^{\nu b} dx^b + \frac{1}{4} dx^\nu v_{\sigma a} v^{\sigma a} \right). \tag{16}$$

In four dimensions we can also consider a tensor valued 2-form [29]

$$\Omega_2^{\rho\nu\lambda} = L^\rho \tilde{L}^{(\lambda\nu)} = dx^\rho \left(d\tilde{c}^{(\lambda\nu)} + \frac{1}{2} \epsilon_{ab} v^{(\lambda a} dv^{\nu)b} \right), \tag{17}$$

which allows us to construct an action for an exotic string. If we eliminate the closed 2-form $dx^\rho d\tilde{c}^{\mu\nu}$ and we add the antisymmetric part in λ, ν , which is a closed form, we get

$$\Omega_2^{\rho\nu\lambda} = dx^\rho v^{\lambda a} dv^{\nu b} \epsilon_{ab}. \quad (18)$$

Notice that

$$\Omega_3^{\rho\nu\lambda} = d(\Omega_2^{\rho\nu\lambda}) = L^\rho L^{\lambda a} L^{b\nu} \epsilon_{ab} \quad (19)$$

is a closed invariant 3-form. Since $\Omega_2^{\rho\nu\lambda}$ is not an invariant 2-form of the unextended string Galilei algebra we conclude that the Eilenberg-Chevalley cohomology of degree three is not trivial.

The string action is obtained by taking a linear combination of the pullback of the previous 2-forms on the world sheet. T is the string tension and $\theta_{\rho\nu\lambda}$ is a spurion tensor field that generalizes the magnetic field appearing in the case of a particle in a magnetic field in $2+1$ dimensions.

$$S = T \int d^2\sigma \det[e] \left(v^{\mu a} e_\mu^j \partial_j x^a + \frac{1}{2} v_{\mu a} v^{\mu a} \right) + \theta_{\rho\nu\lambda} \int d^2\sigma \det[e] \epsilon^{\rho\mu} \left(e_\mu^j v^{\lambda a} \partial_j v^{\nu b} \epsilon^{ab} \right). \quad (20)$$

Here $\partial_i = (\partial/\partial\tau, \partial/\partial\sigma)$. The metric tensor is as before $\eta^{\mu\nu} = (-1, +1)$. The $e_i^\mu = \partial_i x^\mu$ are the zweibein, e_μ^j are the inverse zweibein, and $\det[e] = \det[e_i^\mu]$ is the corresponding determinant. Notice that $[T] = \ell^{-2}$ and $[\theta] = \ell^{-1}$. In the previous equation $\theta_{\mu\rho\lambda}$ is a spurion tensor.

If we want to get by double dimensional reduction the action for the exotic particle [7, 8, 9, 10, 11, 12, 13, 14, 15, 16] we have to make a particular choice for the parameters $\theta_{\rho\nu\lambda}$. By performing the dimensional reduction of the exotic term in eq. (20) we find

$$\theta_{\rho\nu\lambda} \int d\tau \epsilon^{\rho 0} v^{\lambda a} \partial_\tau v^{\nu b} \epsilon_{ab}, \quad (21)$$

to be compared with the exotic term of the action for the particle

$$\kappa \int d\tau v^{0a} \partial_\tau v^{0b} \epsilon_{ab}. \quad (22)$$

Therefore the unique choice is to take

$$\theta_{\rho\nu\lambda} = \theta_\lambda \epsilon_{\rho\nu}, \quad (23)$$

with $\theta_1 = 0$. Other possible forms of the spurion field will not be discussed in this paper.

IV. DYNAMICS OF THE EXOTIC STRING

In this section we will study the dynamics of the exotic string that under dimensional reduction produces the exotic particle. We will use eq. (23) without the restriction on θ_μ . In the static gauge, where we make the identification $(x^0, x^1) = (\tau, \sigma)$, the complete string action becomes

$$S = T \int dx^0 dx^1 \left(v_{\mu a} \partial^\mu x^a + \frac{1}{2} v_{\mu a} v^{\mu a} \right) + \theta^\lambda \int dx^0 dx^1 \left(\eta^{\mu\nu} \partial_\nu v_{\mu b} v_{\lambda a} \epsilon^{ab} \right). \quad (24)$$

The Lagrangian equations of motion are

$$T (-\partial_\mu x_a + v_{\mu a}) + \theta_\mu \partial^\sigma v_{\sigma b} \epsilon^{ab} - \theta^\lambda \partial_\mu v_{\lambda b} \epsilon^{ba} = 0, \quad (25)$$

$$\partial^\mu v_{\mu a} = 0. \quad (26)$$

The boundary conditions in general imply

$$(-T v_{1a} \delta x^a + \theta^\lambda v_{\lambda a} \epsilon^{ab} \delta v_{1b})|_0^L = 0, \quad (27)$$

where L is the length of the string.

These conditions can be satisfied, for example, in the following ways

1. By choosing both x^a and $v^{\lambda a}$ as periodic functions.
2. By choosing x^a periodic and $v^{\lambda a}$ with fixed values at the boundaries.
3. By choosing periodic $v^{\lambda a}$ and x^a with fixed values at the boundaries.
4. By choosing both x^a and $v^{\lambda a}$ with fixed values at the boundaries.
5. By choosing $v^{\lambda a} = 0$ at the boundaries.

Notice that the cases 1) and 5) correspond to a closed and an open string respectively (case $\theta_\mu = 0$). Notice too that, for $\theta_\mu = 0$, one can eliminate the variables v , which are essentially the derivative of the variable x .

In order to understand which are the physical degrees of freedom of the model and their dynamics we will now study the hamiltonian formalism.

Let us start by evaluating the canonical momenta:

$$\Pi_{\mu a} = \frac{\partial \mathcal{L}}{\partial \dot{v}^{\mu a}}, \quad p_a = \frac{\partial \mathcal{L}}{\partial \dot{x}^a}, \quad (28)$$

One finds the following primary constraints:

$$\begin{aligned}\phi_a &= \Pi_{0a} + \theta^\mu v_{\mu b} \epsilon^{ab} = 0, \\ \chi_a &= \Pi_{1a} = 0, \\ \xi_a &= \frac{1}{\sqrt{T}} (p_a + T v_{0a}) = 0,\end{aligned}\tag{29}$$

where $\Pi_{\mu a}$ are the momenta conjugated to $v_{\mu a}$ and P_a the ones conjugated to x_a , where, for a generic field ϕ , $\dot{\phi} = \partial\phi/\partial t = \partial\phi\partial x^0$, with $x^0 = t$.

The canonical Hamiltonian density turns out to be:

$$\mathcal{H}_c = p_a \dot{x}_a + \Pi_{\mu a} \dot{v}^{\mu a} - \mathcal{L} = \frac{1}{2} T (v_{0a}^2 - v_{1a}^2) - T v_{1a} x'_a - \theta^0 v_{0a} v'_{1b} \epsilon_{ab} - \theta^1 v_{1a} v'_{1b} \epsilon^{ab},\tag{30}$$

where a prime denotes derivative with respect to σ (or x_1).

It is easily verified that the following combinations of primary constraints

$$K_a = \frac{(-1)^a}{\sqrt{T}} (+\theta_1 \epsilon_{ab} \xi_b + \sqrt{T} \chi_a)\tag{31}$$

are first class. By taking the independent combinations

$$S_a = \frac{(-1)^a}{z} (\sqrt{T} \epsilon_{ab} \xi_b - \theta_1 \chi_a), \quad z^2 = T + \theta_1^2,\tag{32}$$

one sees that the primary constraints separate as the first class, K_a , and the second class ones (ϕ_a, S_a) .

The matrix of the Poisson brackets among the second class constraints, (ϕ_a, S_a) , is given by:

$$\begin{array}{c|cccc} & \phi_2 & \phi_3 & S_2 & S_3 \\ \hline \phi_2 & 0 & -2\theta^0 & 0 & z \\ \phi_3 & 2\theta^0 & 0 & z & 0 \\ S_2 & 0 & -z & 0 & 0 \\ S_3 & -z & 0 & 0 & 0 \end{array} \times \delta(\sigma - \sigma').\tag{33}$$

The determinant of the σ, σ' independent part of this matrix turns out to be proportional to z^4 . We can now evaluate the Dirac brackets with respect to these constraints. Remember that the general form for the Dirac brackets, given a set (ψ_α) of constraints, with a Poisson bracket matrix given by $C_{\alpha,\beta}$, is

$$\{A, B\}^* = \{A, B\} - \sum_{\alpha,\beta} \{A, \psi_\alpha\} C_{\alpha,\beta}^{-1} \{\psi_\beta, B\}.\tag{34}$$

Just to give an example, considering the set (ϕ_a, S_a) , we find the following result

$$\begin{aligned}
\{x_a(\sigma, \tau), x_b(\sigma', \tau)\}^* &= -\frac{2\theta^0}{z^4}\epsilon_{ab}\delta(\sigma - \sigma'), \\
\{v_{\mu a}(\sigma, \tau), v_{\nu b}(\sigma', \tau)\}^* &= -\frac{\theta_1}{z^2}\epsilon_{ab}\left[(\eta_{\mu 0}\eta_{\nu 1} + \eta_{\mu 1}\eta_{\nu 0}) + \frac{2\theta^0\theta^1}{z^2}\eta_{\mu 1}\eta_{\nu 1}\right]\delta(\sigma - \sigma'), \\
\{x_a(\sigma, \tau), v_{\mu b}(\sigma', \tau)\}^* &= \frac{1}{z^2}\delta_{ab}\left[\eta_{\mu 0} + \frac{2\theta^0\theta^1}{z^2}\eta_{\mu 1}\right]\delta(\sigma - \sigma').
\end{aligned} \tag{35}$$

By requiring the stability of the primary constraints, that is by requiring that the Poisson brackets of the primary constraints with the hamiltonian be zero, one finds the following secondary constraints:

$$\lambda_a = T(v_{1b} + x'_b)\epsilon_{ba} + \theta^\mu v'_{\mu a} = 0. \tag{36}$$

With these two constraints the primary first-class constraints (K_a, λ_a) become second-class.

In fact, the Dirac matrix of the constrains

	K_2	K_3	λ_2	λ_3	
K_2	0	0	0	$-T$	
K_3	0	0	$-T$	0	$\times \delta(\sigma - \sigma')$
λ_2	0	T	0	$-\frac{2T^2\theta^0\theta_1^2}{z^4}$	
λ_3	T	0	$\frac{2T^2\theta^0\theta_1^2}{z^4}$	0	

(37)

has a determinant proportional to $T^4 \neq 0$. We now iterate the procedure to evaluate the final Dirac brackets, including the new set of second class constraints. The final result is (we write only the non vanishing Dirac brackets)

$$\begin{aligned}
\{x_a(\sigma, \tau), x_b(\sigma', \tau)\}^{**} &= -\frac{2\theta^0}{T^2}\epsilon_{ab}\delta(\sigma - \sigma'), \\
\{x_a(\sigma, \tau), v_{0b}(\sigma', \tau)\}^{**} &= -\frac{1}{T}\delta_{ab}\delta(\sigma - \sigma'),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\{x_a(\sigma, \tau), p_b(\sigma', \tau)\}^{**} &= \delta_{ab}\delta(\sigma - \sigma'), \\
\{x_a(\sigma, \tau), \Pi_{0b}(\sigma', \tau)\}^{**} &= \frac{\theta^0}{T}\epsilon_{ab}\delta(\sigma - \sigma'),
\end{aligned} \tag{39}$$

with the ** brackets evaluated as in (34), using the inverse of the matrix (37). At this point we can use the full set of constraints and the final phase space can be taken as (x_a, p_a) .

Notice that using the constraints λ_a we can form the combination

$$v_{12}\lambda_3 - \lambda_2 v_{13} = T v_{1a}^2 + T v_{1a} x'_a - \theta^0 v'_{0a} v_{1b} \epsilon^{ab} - \theta^1 v_{1a} v'_{1b} \epsilon^{ab} = 0. \tag{40}$$

By taking the canonical hamiltonian, see equation(30)

$$H_c = \int d\sigma \mathcal{H}_c, \quad (41)$$

we see that the last three terms of the previous identity coincide with the last three terms in H_c , except for a sign and a total derivative in σ , containing only the $v^{\mu a}$ variables. This total derivative is zero for periodic boundary conditions and it is a constant for fixed boundary conditions. Therefore, apart from this constant, we get

$$H_c = \frac{T}{2} \int d\sigma \left(v_{0a}^2 + v_{1a}^2 \right). \quad (42)$$

V. EQUIVALENCE WITH THE FREE NON-RELATIVISTIC STRING

In order to express the hamiltonian in terms of the independent variables x^a and p_a one has to solve the constraints ξ_a and λ_a . Since these constraints in general depend on v'_{1a} , the hamiltonian density that one gets is generally non-local. See appendix A for the expression in terms of x^a, p_a . However, we can show that this non-locality can be eliminated by a convenient change of variables. Let us start considering the simpler case $\theta_1 = 0$, where the constraints do not depend on v'_{1a} . By solving the constraints we get

$$v_{12} = -x'_2 - \frac{\theta^0}{T^2} p'_3, \quad v_{13} = -x'_3 + \frac{\theta^0}{T^2} p'_2. \quad (43)$$

Therefore the hamiltonian is given by

$$H_c = \int d\sigma \left[\frac{1}{2T} p_a^2 + \frac{1}{2} \frac{\theta^{02}}{T^3} p_a'^2 + \frac{T}{2} x_a'^2 + \frac{\theta^0}{T} \epsilon^{ab} x'_a p'_b \right]. \quad (44)$$

From (43) we see that defining

$$y_2 = x_2 + \frac{\theta^0}{T^2} p_3, \quad y_3 = x_3 - \frac{\theta^0}{T^2} p_2, \quad (45)$$

the hamiltonian can be written as

$$H_c = \int d\sigma \left(\frac{p_a^2}{2T} + \frac{T}{2} y_a'^2 \right). \quad (46)$$

Since the variables v_{1a} (and correspondingly y_a) have zero Dirac brackets with themselves, the theory is equivalent to a non relativistic free string [27].

Coming back to the action in eq. (24) for the general case $\theta_1 \neq 0$, if we integrate by parts the last term (neglecting again the total derivative for the reasons explained above) we get

$$S = T \int dx^0 dx^1 \left[v_{\mu a} \left(-\partial^\mu x^a - \frac{\theta^\lambda}{T} \partial^\mu v_{\lambda b} \epsilon^{ba} \right) + \frac{1}{2} v_{\mu a} v^{\mu a} \right], \quad (47)$$

By defining now the new variables

$$y^a = x^a - \frac{\theta^\lambda}{T} v_{\lambda b} \epsilon^{ba}, \quad (48)$$

we see that the model is equivalent to the free non relativistic string [23] also for $\theta^1 \neq 0$, since the new action coincides with the first term in eq. (20). Also in this case the Dirac brackets of the y'_a 's vanish making the theory a commutative one.

However we have to consider the boundary conditions discussed in Section IV. When x_a and y_a satisfy the same boundary conditions, as for instance in cases 1), 4) and 5), there is a complete equivalence of this exotic string with the ordinary one. In the cases 2) and 3) the y_a variables do not satisfy definite boundary conditions and we loose the equivalence. Notice also that, when the equivalence is realized, the invariance group (in the y_a variables) is isomorphic to the extended Galilei group without the extended generator $\tilde{Z}_{(\mu\nu)}$.

VI. DISCUSSION

In this paper we have constructed an exotic classical non-relativistic string in 3+1 dimensions. The model contains, apart from the ordinary non-relativistic action, an extra term with a spurion field that exists only in 3+1 dimensions. The existence of this extra term is associated to the existence of a new term in the extended Galilei algebra in 4 dimensions. In general the model is non-commutative in the sense that the Dirac bracket of the physical coordinates are not vanishing. It is possible to introduce a new set of coordinates in which the model becomes commutative if the boundary conditions of the old and the new coordinates coincide. We should point out that the transformation properties of the old and the new coordinates are different with respect to the Galilei transformations.

This exotic string by double dimensional reduction reproduces the exotic Galilei particle in 2+1 dimensions.

It should be noticed that the model considered in this paper could be obtained from an exotic relativistic model in 4 dimensions as non-relativistic limit.

We conclude by saying that it will be interesting to see if exists an exotic string theory for a non-flat case, in such a way that under double dimensional reduction it could reproduce the exotic Newton-Hooke particle in 2+1 dimensions.

APPENDIX A: THE HAMILTONIAN DENSITY IN THE REDUCED CANONICAL SPACE

From the constraints λ_a and ξ_a the variables $v_{\mu a}$ can be obtained as a function of the canonical coordinates of the reduced space $\{x^a, p_a\}$. To this end we must solve the following differential system

$$v'_{12}(\sigma) = \omega v_{13}(\sigma) + f_1(\sigma), \quad v'_{13}(\sigma) = -\omega v_{12}(\sigma) + f_2(\sigma), \quad (\text{A1})$$

where

$$f_1 = \omega(x'_3 - \frac{\theta_0}{T^2}p'_2), \quad f_2 = -\omega(x'_2 + \frac{\theta_0}{T^2}p'_3). \quad (\text{A2})$$

The other variables v_{0a} are easily obtained from the constraints ξ_a . In these equations we have put $\omega = T/\theta^1$. Observe that ω becomes infinite when θ^1 is zero. In this limit the system degenerates to an algebraic system, which was already discussed in Section V. The system (A1) can be easily integrated and the solution is

$$v_{12} = \frac{1}{\sqrt{2}}(w_1 + w_2), \quad v_{13}(\sigma) = \frac{i}{\sqrt{2}}(w_1 - w_2), \quad (\text{A3})$$

where

$$w_1(\sigma) = \frac{1}{\sqrt{2}}e^{i\omega\sigma} \left(\int_0^\sigma e^{-i\omega\sigma'} (f_1(\sigma') - if_2(\sigma')) d\sigma' + d \right), \quad w_2(\sigma) = w_1(\sigma)^*. \quad (\text{A4})$$

In these equations the complex constant of integration d is determined by the boundary conditions discussed in Section IV. With these expressions one can evaluate the canonical Hamiltonian H_c (see eq. (42)). The resulting hamiltonian is manifestly non-local in the variables (x^a, p_a) .

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- [1] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance*. In: *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972).
 - [2] A. Ballesteros, M. Gadella and M. del Olmo, *Moyal quantization of 2+1 dimensional Galilean systems*. *Journ. Math. Phys.* **33** (1992) 3379;
 - [3] Y. Brihaye, C. Gónzalez, S. Giller and P. Kosiński, *Galilean invariance in 2 + 1 dimensions*. arXiv:hep-th/9503046.
 - [4] V. Bargmann, “On Unitary Ray Representations Of Continuous Groups,” *Annals Math.* **59** (1954) 1.
 - [5] J.-M. Lévy-Leblond, “Group-Theoretical Foundations of Classical Mechanics: The Lagrangian Gauge Problem,” *Comm. Math. Phys.* **12** (1969) 64.
 - [6] J. A. de Azcarraga and J. M. Izquierdo, “Lie groups, Lie algebras, cohomology and some applications in physics.” Cambridge Univ. Press, 1995.
 - [7] J. Lukierski, P. C. Stichel and W. J. Zakrzewski, “Galilean-invariant (2+1)-dimensional models with a Chern-Simons-like term and $D = 2$ noncommutative geometry,” *Annals Phys.* **260** (1997) 224 [arXiv:hep-th/9612017].
 - [8] C. Duval and P. A. Horvathy, “The “Peierls substitution” and the exotic Galilei group,” *Phys. Lett. B* **479** (2000) 284 [arXiv:hep-th/0002233]; “Exotic Galilean symmetry in the noncommutative plane, and the Hall effect,” *J. Phys. A* **34** (2001) 10097 [arXiv:hep-th/0106089].
 - [9] R. Jackiw and V. P. Nair, “Anyon spin and the exotic central extension of the planar Galilei group,” *Phys. Lett. B* **480** (2000) 237 [arXiv:hep-th/0003130].
 - [10] C. Duval, Z. Horvath and P. A. Horvathy, “Exotic plasma as classical Hall liquid,” *Int. J. Mod. Phys. B* **15** (2001) 3397 [arXiv:cond-mat/0101449].
 - [11] P. A. Horvathy and M. S. Plyushchay, “Non-relativistic anyons, exotic Galilean symmetry and noncommutative plane,” *JHEP* **0206** (2002) 033 [arXiv:hep-th/0201228].
 - [12] P. A. Horvathy and M. S. Plyushchay, “Anyon wave equations and the noncommutative plane,” *Phys. Lett. B* **595** (2004) 547 [arXiv:hep-th/0404137].

- [13] P. A. Horvathy and M. S. Plyushchay, “*Nonrelativistic anyons in external electromagnetic field,*” Nucl. Phys. B **714** (2005) 269 [arXiv:hep-th/0502040].
- [14] J. Negro, M. A. del Olmo and J. Tosiek, “*Anyons, group theory and planar physics,*” J. Math. Phys. **47** (2006) 033508 [arXiv:math-ph/0512007].
- [15] M. A. del Olmo and M. S. Plyushchay, “*Electric Chern-Simons Term, Enlarged Exotic Galilei Symmetry And Noncommutative Plane,*” Annals Phys. **321** (2006) 2830 [arXiv:hep-th/0508020].
- [16] P. A. Horvathy, “*Non-commutative mechanics, in mathematical and in condensed matter physics,*” arXiv:cond-mat/0609571.
- [17] A. P. Polychronakos, “Noncommutative Fluids,” arXiv:0706.1095 [hep-th].
- [18] O. Arratia, M. A. Martin and M. A. Olmo, “*Classical Systems and Representation of (2+1) Newton-Hooke Symmetries,*” arXiv:math-ph/9903013.
- [19] P. D. Alvarez, J. Gomis, K. Kamimura and M. S. Plyushchay, “(2+1)D exotic Newton-Hooke symmetry, duality and projective phase,” arXiv:hep-th/0702014.
- [20] J. Gomis and H. Ooguri, “Non-relativistic closed string theory,” J. Math. Phys. **42** (2001) 3127 [arXiv:hep-th/0009181].
- [21] U. H. Danielsson, A. Guijosa and M. Kruczenski, “IIA/B, wound and wrapped,” JHEP **0010** (2000) 020 [arXiv:hep-th/0009182].
- [22] S. R. Coleman, J. Wess and B. Zumino, “*Structure of phenomenological Lagrangians. 1,*” Phys. Rev. **177** (1969) 2239; C. G. . Callan, S. R. Coleman, J. Wess and B. Zumino, “*Structure of phenomenological Lagrangians. 2,*” Phys. Rev. **177** (1969) 2247.
- [23] J. Bruges, T. Curtright, J. Gomis and L. Mezincescu, “Non-relativistic strings and branes as non-linear realizations of Galilei [arXiv:hep-th/0404175].
- [24] J. Bruges, J. Gomis and K. Kamimura, “Newton-Hooke algebras, non-relativistic branes and generalized pp-wave Phys. Rev. D **73** (2006) 085011 [arXiv:hep-th/0603023].
- [25] J. Gomis, K. Kamimura and P. West, “*The construction of brane and superbrane actions using non-linear realisations,*” Class. Quant. Grav. **23** (2006) 7369 [arXiv:hep-th/0607057].
- [26] J. P. Gauntlett, J. Gomis and P. K. Townsend, “Particle Actions As Wess-Zumino Terms For Space-Time (Super)Symmetry Groups,” Phys. Lett. B **249** (1990) 255.
- [27] J. Gomis and F. Passerini, “Rotating solutions of non-relativistic string theory,” Phys. Lett. B **617** (2005) 182 [arXiv:hep-th/0411195].

- [28] The introduction of a Goldstone field associated to a central extension was first considered in [26]
- [29] We could have considered also a scalar 2-form $\tilde{\Omega} = L_z \eta_{\mu\nu} \tilde{L}^{(\lambda\nu)}$, however under double dimensional reduction it does not produce the action of an exotic particle